

NUMERICAL MODELING AND ANALYSIS OF MACHINE NOISE IN A METAL-WORK WORKSHOP

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Abstract: Noise is often used to describe unwanted sound (or generally, vibration). The study of Noise and its effects is closely connected with the fields of Vibration, Acoustics and Sound thus lending itself to a fitting acronym, Vibration Acoustic Noise and Sound (VANS) that best characterizes this inter-related field of study. Finite Element Method, which is one of the numerous numerical tools, is used to divide the solution domain into simply shaped regions or “elements” after which the final solution is generated using an assembly of the individual solutions in a systematic manner that ensures continuity at the inter-element boundaries. Historically FEM was the first element to be used for acoustics, however, in recent times numerous works tend to favour another type of numerical tool known as the Boundary Element Method (Direct and Indirect method) in noise analysis especially for its ability to solve both interior and exterior problems compared to the Finite Element Method which is regarded as suited only for classic interior/cavity problems. But contrary to FEM’s perceived limitation, the major difference between the two methods is man-power related not machine-related. Thus, owing to its rather compelling merits such as a banded and symmetric system of equations which favour computability, this work considers the Finite Element approach to noise analysis in a Metal-work Workshop. The problem is considered as dynamic thus the Modal Analysis technique is used to analyze the natural frequencies (eigenvalues) within the domain into four normal (4) modes (eigenvectors). A percentage error of about 2% error is realized using two linear triangular elements to discretize the domain.

Keywords: Noise; Sound Pressure Level; Eigenvectors; Galerkin Finite Element Method; Direct and Modal Frequency Analysis

1. INTRODUCTION

The study of *Vibration* is concerned with the oscillatory motion of bodies and the forces associated with them. All bodies possessing mass and elasticity are capable of vibration [1].

Sound is defined as a pressure disturbance that moves through a material at a speed which is depended on the material [2].

Acoustics refers to the study of sound, namely its production, transmission through solid and fluid media, and any other phenomenon engendered by its propagation through media [3]. Acoustics is concerned with the generation and space-time evolution of small mechanical perturbations in a fluid (sound waves) or in a solid (elastic waves). Then equations of acoustics are simply obtained by linearization of the equations of the mechanics of continua [4].

Noise is usually defined as any perceived sound that is objectionable or damaging for a human [5]. Balachandran and Magrab [6] refer to Noise as vibrations which are responsible for unpleasant sounds.

From the above definitions, it is obvious that these four concepts share similar characteristics and can be listed in the following order as: *Vibration*, *Acoustics*, *Noise* and *Sound*, thus inspiring the authors’ adoption of the acronym ‘VANS’ to describe this particular field of multi-disciplinary study.

Finite Element Method is a numerical prediction technique which involves finding the distribution of one (or several) field variable(s) in a continuum domain, governed by an appropriate (set of) partial differential equation(s) and boundary conditions [7]. It is also referred to as *Finite Element Analysis* (FEA) [8]. Generally, analyses carried out by applying FEA to dynamic problems, such as in this case, often involve these two categories [9]:

- 1, Direct Frequency-Response Analysis and
2. Modal Frequency-Response Analysis

A couple of previous work serve as precursor to this present work namely Olanipekun [10] and Layeni and Olanipekun [11], where the former involved calculation of room reverberation-constant by using the Energy Method among other analyses and control procedures carried out in a typical Metal-work Workshop environment, while the latter, however, emphasized more on the analysis of the noise/sound level measurement in a Metal-work Workshop where the sound level measurements were conducted. In both cases, the results obtained were consistent with literature on Noise control and analysis.

Unlike these previous works aforementioned, the present work distinctly applies Finite Element Method, using the Galerkin Weighted Residual Method for the integral Formulation, as the primary tool for Noise Analysis of the same workshop space. In particular, of the two techniques used in finite element analysis of this nature as outlined in the foregoing, the *Modal Analysis (Mode Superposition Technique)* has been employed in this research.

It has been seen that both techniques are necessary for a complete and comprehensive analyses of the system [12]. While the former involves solving a system of equation much in the same form as applied to displacement of bar element under strain measurements and temperature distribution in heat application [7], the latter differs from former's linear static pressure approach by considering the dynamic response of continuous media (sound) from a frequency domain (eigenvalues) view-point [8] [9] [13].

2. METHODOLOGY

The governing equation for sound and noise is the *Acoustic Wave Equation* [14, 15]

$$\nabla^2 p = \frac{1}{c^2} \frac{\partial^2 p}{\partial t^2} \quad (2.1)$$

Where,

∇^2 represents the Laplacian operator [16], that is,

$$\nabla^2 = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \right) \quad (2.2)$$

$p(x, y, z)$ is the *acoustic perturbation pressure*

c is the *velocity* of propagation of the wave

This is a second order differential equation that is linearized for simplicity. The one dimensional case is given as:

$$\frac{\partial^2 p'}{\partial x^2} = \frac{1}{c^2} \frac{\partial^2 p}{\partial t^2} \quad (2.3)$$

This equation is known as the linearized *Acoustic Wave Equation*.

The Constitutive Equations are

$$\tilde{p} = c^2 \tilde{\rho} \quad (2.4)$$

Introducing the *velocity potential* $\phi(x, y, z)$, the Wave Equation may be presented in the following form:

$$\nabla^2 \phi - \frac{1}{c^2} \frac{\partial^2 \phi}{\partial t^2} = 0 \quad (2.5)$$

With a pair of imposed *Robin Boundary Conditions* (that is, one *Dirichlet* and the other *Neumann* boundary conditions) include:

$$p' = -\rho_0 \frac{\partial \phi}{\partial t} \quad (2.6)$$

$$v' = \nabla \phi$$

This is re-expressed as the *Helmholtz Equation*:

$$\nabla^2 \phi + k^2 \phi = 0 \quad (2.7)$$

Where k is often defined as the *wave number* and is given as:

$$k = \frac{\omega}{c} \quad (2.8)$$

Note that to arrive at the form of the Helmholtz Equation, the velocity potential ϕ with respect to the time harmonic wave is taken as:

$$\phi(x, y, z, t) = \tilde{\phi}(x, y, z)e^{i\omega t} \quad (2.9)$$

So that,

$$\frac{\partial \phi}{\partial t} = (i\omega)\tilde{\phi}e^{i\omega t} = i\omega\phi$$

$$\frac{\partial^2 \phi}{\partial t^2} = (i\omega)^2\tilde{\phi}e^{i\omega t} = -\omega^2\phi$$

Which also implies the following:

$$p' = -i\omega\rho_0\phi = -ikc\rho_0\phi \quad (2.10)$$

Finally, in computing the results, the following basic assumptions were made:

1. The specified boundary surface consist of rigid walls
2. Density of air is uniform at 1.2 kg/m³
3. Speed of sound is constant at 323 m/s at temperature, $T = 22^\circ\text{C}$

2.1. Basic Integral Formulation

In this section, we briefly discuss the derivation of the Finite Element Model for noise, starting from the Galerkin Weighted Residual Method where the *weight function* is the same as the approximation in a bid to minimize the

error as a result of the approximation. Afterwards using the derived integral expression, the Finite Element formulation is obtained.

2.1.1. Galerkin Weighted Residual Method

Firstly, taking the velocity potential form of the Helmholtz equation as the *Residual* function, R , and setting the integral equation equal to zero [18] we have:

$$R = (\nabla^2 \phi + k^2 \phi) \tag{2.11}$$

$$\int_V (\nabla^2 \phi + k^2 \phi) w dV = 0 \tag{2.12}$$

Where, w , is the *weight function*.

Next, Green’s first identity (theorem) is applied to the integral expression where [17] notes that the same is a simple consequence of the divergence theorem.

After considerable mathematical manipulation [18] [19] the following relation is obtained

$$\int_S w \frac{\partial \phi}{\partial n} dS - \int_V \nabla w \nabla \phi dV + \int_V k^2 w \phi dV = 0 \tag{2.13}$$

2.1.2. Finite Element Formulation

The finite element formulation is obtained using a desired finite element which is then followed by the discretization of the domain of interest in this case the workshop hall space. As indicated earlier, the formulation of the finite element is based on the Galerkin weighted residual method where the weight function, w is equal to $\phi(x)$, [20] and is of the general form:

$$\phi(x) = \sum_{j=1}^N \phi_j N_j \tag{2.14}$$

For the acoustic element, a *linear triangular element* (see Figure 1) is chosen whose the shape (or interpolation) functions, N_j , belong to the *Serendipity* interpolation family [8] [14, 15] described in terms of the (x, y) coordinates as follows:

$$N_1(x, y) = 1 - \frac{x}{b} - \frac{y}{h} \tag{a}$$

$$N_2(x, y) = \frac{x}{b} \tag{b}$$

$$N_3(x, y) = \frac{y}{h} \tag{c} \tag{2.15}$$

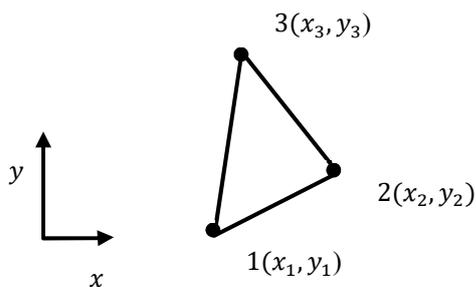


Figure 1: A general three-node triangular element with nodal coordinates

Therefore, the general formulation may be written as:

$$\int_V (\nabla \phi \cdot \nabla N_i - k^2 \phi N_i) dV = \int_S \gamma N_i dS \quad (i = 1, 2 \dots N) \quad (2.16)$$

And finally as:

$$\sum_{j=1}^N \int_V \nabla N_i \cdot \nabla N_j dV \phi_j - k^2 \sum_{j=1}^N \int_V N_i N_j dV \phi_j = \int_S \gamma N_i dS \quad (i = 1, 2 \dots N) \quad (2.17)$$

where, the shape functions from Equation 2.15 have been duly catered for.

2.2. Dynamic/Modal vs. (Pseudo-) Steady-State Analysis

Since the noise/sound propagation is time dependent, hence *transient*, it is often helpful to consider it as a *dynamic* problem rather than a *direct* steady-state one, even though such latter treatment (which is also known as a *propagation* problem) remains valid and can prove useful under certain circumstances.

In such a case, as Bathe [21] duly notes, the goal is therefore to formulate the eigenvalue problem from the finite element equation of motion which is of the following general form:

$$[K]\{\phi\} - k^2[M]\{\phi\} = \{f_f\} \quad (i = 1, 2 \dots N) \quad (2.18)$$

Where,

$[K]$ is the Acoustic Stiffness Matrix

$[M]$ is the *Acoustic Mass Matrix*, which provides a measure of the *compressibility* of the sound pressure [7].

$\{f_f\}$ is the *Nodal Acoustic Flux* (i.e. *forcing function*)

This approach is also called the *Mode Superposition* technique. It is observed that this system of equation is analogous to the equation of motion for a vibrating mass-spring [1] [8]. However, it should be noted that Equation 2.18 is in a simplified form as it does not include the Damping coefficient term $[C]$, neither is a *vibro-acoustics* coupled system approach considered under this present scope. However, subsequent works, for example on the aspect of noise control, etc, will require and therefore incorporate these concepts.

2.3. Discretization and Assembly of System Matrices

The domain (or region) in this case is the workshop hall space which has a quadrilateral shape. Figure 2 shows the hall space of the machine room that is to be analysed.

In compatibility with the geometry of the workshop layout, the rectangular-shaped floor space is discretized into two triangular elements of similar dimensions (with thickness, t), and is illustrated in Figure 3.

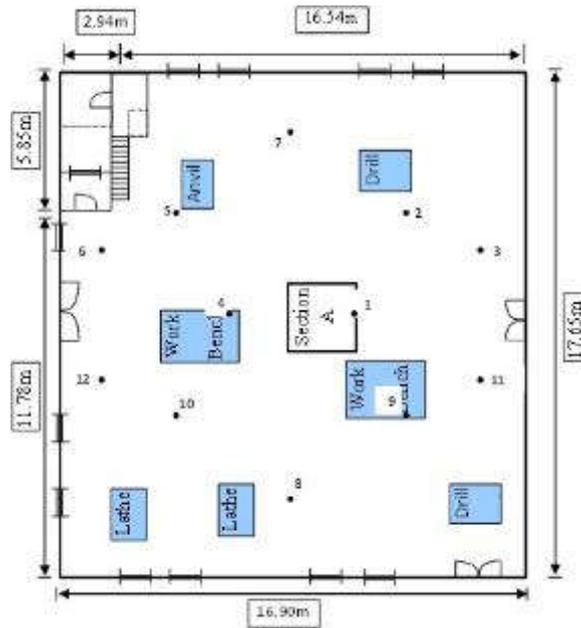


Figure 2: Floor Plan Layout of Metal-work Section A of the Workshop

Note that this triangular element used here is that of a *constant velocity element* [14] similar to a *constant strain element* employed in equilibrium problems regarding finite element strain-displacement relations for rigid-body structures such as in bar elements, beams, trusses etc. [22].

Also, the arrows indicate the direction of mass flow across the boundary surface of each triangular element [14, 15].

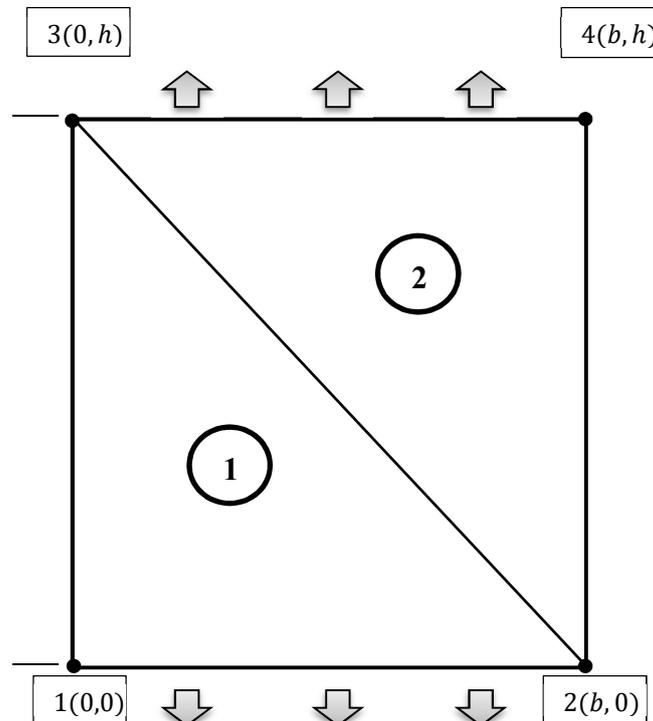


Figure 3: Discretized domain showing the triangular elements with their global node coordinates

2.4. Eigenvalues and Frequencies

There exists an eigenvalue problem of the form:

$$[K]\{\phi\} - k^2[M]\{\phi\} = \{0\} \quad (2.19)$$

Whose *characteristic polynomial/equation* of n degree may be obtained by finding the determinant of the resulting combination of $[K]$ and $[M]$ matrix terms as shown in Equation 2.23.

Basically, the degree n of this polynomial, depends on the number of global nodes of the discretized model which, in this case, $n = 4$. Thus from Equation 2.19, we have the general form of the characteristic equation as:

$$a\lambda^4 + b\lambda^3 + c\lambda^2 + d\lambda + p = 0 \quad (2.20)$$

For which case, the roots of the equations, λ_i , is given as

$$\lambda_i = \frac{k^2bh}{12} \quad (2.21)$$

And from Equation 2.8,

$$k = \frac{\omega}{c}$$

Where, b and h represents the *base* and *height* of the triangular element of arbitrary thickness, t , respectively.

For each root, λ_i , the corresponding natural frequencies ω_i (or f_i , in Hertz (Hz)) which represents the eigenvalues for the particular acoustic system can be obtained thus,

$$f_i = \frac{c}{2\pi} \sqrt{\frac{12\lambda_i}{bh}} \quad (2.22)$$

$$(i = 1, 2, \dots, n)$$

2.5. Eigenvectors and Normal Modes

The corresponding eigenvectors are computed and normalized. It should be noted that computational methods and programming (in this case, MATLAB), were adopted to facilitate results and reduce error, as this process may tend to be rather rigorous mathematically.

The resulting system matrix of eigenvalues for this particular problem was obtained to be:

$$\begin{bmatrix} 2 - 2\lambda & -1 - \lambda & -1 - \lambda & 0 \\ -1 - \lambda & 2 - 8\lambda & -1 + 2\lambda & -5\lambda \\ -1 - \lambda & -1 + 2\lambda & 3 - 4\lambda & -1 + 3\lambda \\ 0 & -5\lambda & -1 + 3\lambda & 1 - 6\lambda \end{bmatrix} \begin{Bmatrix} \Phi_1 \\ \Phi_2 \\ \Phi_3 \\ \Phi_4 \end{Bmatrix} = \{0\} \quad (2.23)$$

3. RESULTS AND DISCUSSION

3.1. Eigenvalues and Frequencies

The following natural circular frequencies were computed with the aid of a computer program implemented in MATLAB:

$$(f_1, f_2, f_3, f_4) = (0, 17.50, 5.38, 11.04)Hz \quad (3.1)$$

Thus, it is observed that these are the values for a typical large hall.

The finite element model (and thus results) outlined here agree in principle with other works, for example when compare with results described by Shuku and Ishihara [23] which also involve using triangular finite element

with cubic interpolation to compute the normal frequencies in a 1.1 by 2.0 (m) rectangular room as well as the interior of a car.

In practice, therefore, the physical implications of these results imply that mechanical/work operations around these principal frequencies should be given utmost attention early at the design stage for noise control.

The percentage error can be obtained from the actual values of frequencies to determine the accuracy of the results for the finite element method procedure. For example, for the second frequency mode, we have:

$$\% \text{ Error} = \frac{f_{2 \text{ numerical}}}{f_{2 \text{ exact}}} \times 100\% \quad (3.2)$$

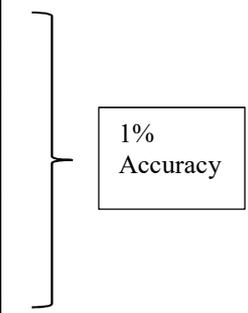
$$\text{Where, } f_{2 \text{ exact}} = \frac{c}{2L} \quad (3.3)$$

And for this particular work, approximately 2% error margin was realized using this two-element mesh, which is a fairly significant result especially considering that an error limit of 10% for numerical methods was predicted by Lord Rayleigh [14, 15].

Notwithstanding, this result can still be improved upon bearing in mind that computational procedures such as these have the capability to minimize error by increasing the number of elements used for discretization although such effort is not without its accompanying trade-offs especially in computational time, data/memory allocation and overall man-power [7] [14, 15] [24, 25].

Table 1 shows a comparison of convergence for two-noded and three-noded line elements done by Anderson [14, 15] indicating how many line elements are needed to get 1% accuracy for the second frequency. By comparison, the results obtained here in this work seem quite satisfactory especially when it is taken into consideration that about 2% error (See Equation 3.2 and 3.3) in the natural frequency was obtained using only two triangular elements.

Table 1: Frequencies and % error of two-noded and three-noded line elements

<i>No.of Elements</i>	<i>Two-noded elements</i>		<i>Three-noded elements</i>	
	<i>f₂, Hz</i>	<i>% error</i>	<i>f₂, Hz</i>	<i>% error</i>
1	18.36	10.3	18.36	10.3
2	18.36	10.3	16.71	0.38
3	17.42	4.6		
4	17.08	2.6		
5	16.92	1.6		
6	16.84	1.2		
7	16.78	0.8		
8	16.74	0.54		
9	16.73	0.48		

3.2. Eigenvectors and Normal modes

The modes have orthogonal mass matrices corresponding to the natural frequencies ($\omega_1, \omega_2, \omega_3, \omega_4$) and two of the eigenvectors (or modes) (i.e. the 1st and 4th modes) are given as:

$$\begin{cases} \phi_1 \\ \phi_2 \\ \phi_3 \\ \phi_4 \end{cases}^1 = \begin{cases} 1.0 \\ 1.0 \\ 1.0 \\ 1.0 \end{cases}$$

$$\begin{cases} \phi_1 \\ \phi_2 \\ \phi_3 \\ \phi_4 \end{cases}^4 = \begin{cases} 0.7849 \\ -0.28475 \\ -0.5491 \\ 0.03724 \end{cases} \tag{3.4}$$

3.3. Mode Shapes

The mode shapes represent deformation patterns across the physical domain [1] [26] [27] [28] which help visualize the dynamic response of the structural assembly, which is represented by the workspace.

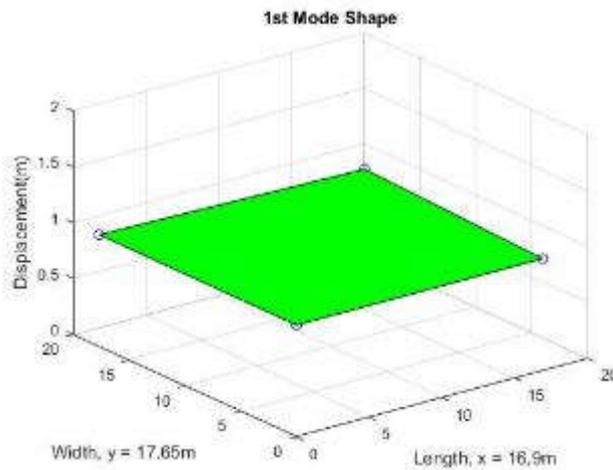


Figure 4: The First Mode of the System (Constant Pressure Mode)

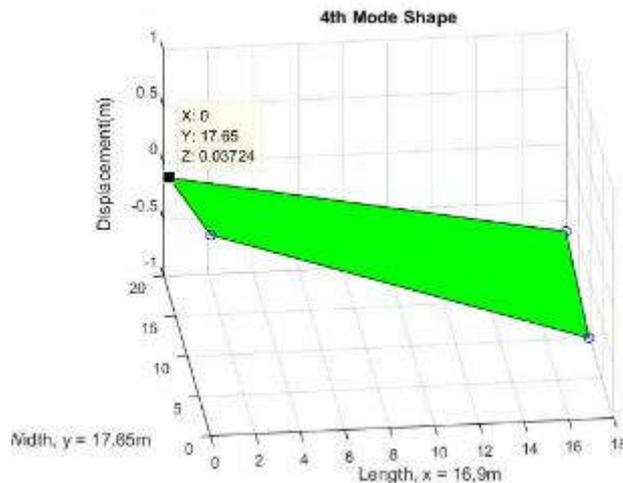


Figure 5: The Fourth Mode Shape of the System

The first mode indicates a constant pressure mode, while the fourth indicates pressure perturbation of the given frequency across the entire hall.

3.4. Noise Abatement Schemes

With this knowledge, the Noise Control Engineer can make well-informed judgment regarding the choice of appropriate design parameters such as damping coefficient, absorption coefficient, etc, that should be incorporated in order to achieve significant overall noise reduction (NR) within the metal-work workshop hall in order to improve sound quality and speech intelligibility for personnel and users of the facility especially during working hours.

Other noise mitigation procedures may also be taken which may include identifying metal-work operations that are most likely to cause noise especially within these *offending* frequencies. Such efforts may entail modifying the methods in which work is being carried out for example lubrication of certain parts of the work-piece and tool piece during machining, adjusting the angle of tool incident on work-piece, providing orientation programmes/seminars that will enlighten users of the workshop on the benefits of embracing noise control practices such as use of hearing protection aids which in turn play a significant role in reducing health risk posed by noise hazards.

An effective, although financially intensive approach will require the implementation of sound insulation and isolation techniques such as constructing sound proof booths for personnel, constructing sound enclosure for noisy machinery, mounting of acoustic tiles [10], designating certain areas above 85 dB(A) as *hearing protection areas* [10] [11], etc. Heavy-duty machinery that transmit mechanical vibration, to structural members (as in coupled systems), for example, floor, wall, etc, may be mounted on dampers to reduce transmissibility loss (TL).

4. CONCLUSION

It has been shown that the natural (fundamental) frequencies and eigenvectors as well as their proper interpretation and application can be successfully conducted using the Finite Element Method and Modal Analysis approach, providing both data and visual representation of the general natural frequency spectrum in a given location, in this case, a metal-work workshop. This first step describes the aspect of the noise analysis after which noise control procedures may then follow.

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